RCWA – RESIDUE-CLASS-WISE AFFINE GROUPS

STEFAN KOHL

ABSTRACT. RCWA is a package for the computer algebra system GAP. It provides implementations of algorithms and methods for computing in certain infinite permutation groups. The RCWA package is not system-dependent and can be used on any machine for which GAP is available. It is distributed free of charge and in source under the terms of the GNU General Public License, and it is available for download at http://www.gap-system.org/Packages/rcwa.html.

1. INTRODUCTION

The GAP [1] package RCWA introduces the class of *residue-class-wise affine* groups, and it provides a large variety of functionality for computing with them. Residue-class-wise affine groups are countable permutation groups acting on the integers or on another 'suitable' ring, whose elements are bijective *residue-class-wise affine* mappings.

For simplicity, in this short description we assume that the underlying ring is $\mathbb{Z}$.

Definition 1.1. A mapping $f : \mathbb{Z} \to \mathbb{Z}$ is called *residue-class-wise affine* if there is a positive integer m such that the restrictions of f to the residue classes (mod m) are all affine, i.e. given by

$$f|_{r(m)} : \; r(m) \to \mathbb{Z}, \quad n \mapsto \frac{a_{r(m)} \cdot n + b_{r(m)}}{c_{r(m)}}$$

for certain coefficients $a_{r(m)}, b_{r(m)}, c_{r(m)} \in \mathbb{Z}$ depending on $r(m)$.

The class of residue-class-wise affine groups includes – but is by far not limited to – the groups of the following types:

- Finite groups, and certain divisible torsion groups which they embed into.
- Free groups of finite rank.
- The modular group $\mathrm{PSL}(2, \mathbb{Z})$.
- Free products of finitely many finite groups.
- Direct products of residue-class-wise affine groups.
- Wreath products of residue-class-wise affine groups with finite groups.
- (Restricted) wreath products of residue-class-wise affine groups with $(\mathbb{Z}, +)$.

Among these groups there are finitely generated groups with unsolvable membership problem and finitely generated groups which are not finitely presented.

The *class transpositions* – these are involutions which interchange two disjoint residue classes – generate a simple group. This simple group has subgroups of all types listed above, and its class of subgroups is closed under the mentioned operations.

Residue-class-wise affine groups have been introduced in the author's thesis [2]. A couple of more recent results including those mentioned above are proved in [3]. Further results obtained with significant help of RCWA can be found in [5] and [6].

2000 *Mathematics Subject Classification.* Primary 20-04, Secondary 20B22, 20B40, 11-04, 11Y55, 11B99.

2. Mathematical Functionality

The mathematical functionality of RCWA comprises roughly the following:

- Testing whether a given residue-class-wise affine mapping is surjective respectively injective. Computing images and preimages of set-theoretic unions of residue classes under such mappings. Computing cycles and trajectories.
- Performing arithmetical operations with residue-class-wise affine mappings: multiplication, inversion and the extraction of roots.
- Computing the order of a bijective residue-class-wise affine mapping.
- Determining a number of attributes and properties of a given residue-class-wise affine mapping, and computing certain related transition graphs and -matrices.
- Factoring a residue-class-wise affine permutation into involutions which interchange disjoint residue classes and permutations with only one non-identity affine partial mapping. This is done using a rather complicated heuristic method.

  It is so far an open problem whether any residue-class-wise affine permutation has such a factorization. Anyway, the routine has successfully been used to factor a permutation which has already been investigated by Lothar Collatz in 1932 and whose cycle structure is unknown so far.
- Determining a couple of basic attributes and properties of residue-class-wise affine groups. For example the order can be computed, and it can be checked whether a group is *tame*, i.e. whether the number of distinct affine partial mappings of its elements is bounded. Also certain finite homomorphic images can be computed, and sometimes it can be checked whether a group is solvable, perfect resp. simple.
- Computing a string which describes the structure of a given residue-class-wise affine group to a certain extent. Sections on whose structure RCWA cannot obtain information automatically are denoted by `<unknown>`.
- Solving the membership problem, i.e. testing whether a given residue-class-wise affine permutation lies in a residue-class-wise affine group given by generators. This works quite often – but not always, as there are residue-class-wise affine groups with unsolvable membership problem.
- Factoring a residue-class-wise affine permutation into the generators of a given residue-class-wise affine group.
- Testing whether a subgroup of a residue-class-wise affine group is normal. Sometimes also its index and the factor group can be computed.
- Forming direct products of residue-class-wise affine groups.
- Forming wreath products of residue-class-wise affine groups with finite permutation groups and with the infinite cyclic group $(\mathbb{Z}, +)$.
- Determining faithful representations of free products of finite groups as residue-class-wise affine groups.
- Testing whether a given residue-class-wise affine group acts transitively on $\mathbb{Z}$. Testing for multiple transitivity is in many cases possible in an interactive session with a sequence of commands depending on the particular group.
- Computing an element of a given residue-class-wise affine group which maps a given tuple of integers to a given other tuple of integers.
- Determining matrix representations of some of those residue-class-wise affine groups which have faithful finite-dimensional linear representations.

Items which would need longer explanations have been omitted from this list. Descriptions of many of the algorithms and methods implemented in RCWA can be found in [4].

3. AIMS AND SCOPE

Residue-class-wise affine groups are still far from being understood very well. This package is intended to serve as a tool for obtaining a better understanding of their rich and interesting group theoretical and combinatorial structure.

Results which the author has already obtained with substancial help of this package can be found in [2], [5], [6] and in [3].

This package can be applied in various ways to various different problems, and it is just not possible to say what can be found out with its help about which groups. The best way to get an idea about this is likely to experiment with the examples discussed in the manual.

Of course the RCWA package often does not provide an out-of-the-box solution for a given problem. However, quite often it is possible to find an answer for a given question by using an interactive trial-and-error approach.

4. NEEDED SOFTWARE

In its version 2.5.4, the package RCWA needs at least GAP 4.4.7. Further it needs the GAP packages ResClasses (version $\geqslant$ 2.5.1), GRAPE (version $\geqslant$ 4.0), Polycyclic (version $\geqslant$ 2.1) and GAPDoc (version $\geqslant$ 1.0).

All needed packages are already present in an up-to-date standard GAP installation. Therefore installing GAP is usually the only thing one needs to do in order to be able to use RCWA .

5. AVAILABLE DOCUMENTATION

The manual of RCWA is available in the formats PDF and HTML. Further the documentation can be accessed during a session via GAP 's online help system. In total, i.e. including the discussion of many examples, the manual comprises about 100 pages. In order to use the package, it is of course by far not necessary to read the manual cover-to-cover.

REFERENCES

1. The GAP Group, *GAP – Groups, Algorithms, and Programming; Version 4.4.10*, 2007, http://www.gap-system.org.
2. Stefan Kohl, *Restklassenweise affine Gruppen*, Dissertation, Universität Stuttgart, 2005, available for download at http://elib.uni-stuttgart.de/opus/volltexte/2005/2448/.
3. ———, *A simple group generated by involutions interchanging residue classes of the integers*, (2006), approx. 11 pages, preprint, available at http://www.cip.mathematik.uni-stuttgart.de/~kohlsn/preprints/simplegp.pdf.
4. ———, *Algorithms for a class of infinite permutation groups*, (2007), approx. 36 pages, preprint, available at http://www.cip.mathematik.uni-stuttgart.de/~kohlsn/preprints/compute.pdf.
5. ———, *Wildness of iteration of certain residue-class-wise affine mappings*, Adv. in Appl. Math. **39** (2007), no. 3, 322–328, doi:10.1016/j.aam.2006.08.003.
6. ———, *On conjugates of Collatz-type mappings*, Int. J. Number Theory **?** (2008), no. ?, approx. 4 pages, to appear.

INSTITUT FÜR GEOMETRIE UND TOPOLOGIE, PFAFFENWALDRING 57, UNIVERSITÄT STUTTGART
70550 STUTTGART, GERMANY
E-mail address: kohl@mathematik.uni-stuttgart.de