

CASA

Computer Algebra Software for Algebraic Geometry

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1 Scope of CASA

CASA¹ is a special-purpose system for computational algebra and constructive algebraic geometry.

Constructive methods in algebra and algebraic geometry have been gaining more and more importance with the availability of computers and computer algebra software. Since its first version, CASA has been designed to perform computations and utilize reasoning about algebraic and geometrical objects in classical affine and projective spaces over algebraically closed fields of characteristic zero. More precisely, the field has to be a computable field in the sense of the underlying computer algebra system Maple, i. e. all the arithmetic operations have to be available in the system. Usually, the field of computation is the rational numbers \mathbf{Q} or a finite algebraic extension thereof.

2 System Requirements

CASA is built on the kernel of Maple and is fully independent of the operating system. Hence, it can be used on every hardware where Maple is running. The current version of CASA (V2.5) requires Maple V.5.

3 Availability

The system can be obtained by anonymous ftp at <ftp.risc.uni-linz.ac.at>. Major releases will be located in the directory `/pub/CASA`. Bug correction and minor updates will be put in `/pub/CASA/update`. The CASA home page on the World Wide Web is <http://www.risc.uni-linz.ac.at/software/casa>.

The system is freely distributed under the following conditions: any research activity which uses CASA should cite the authors and the system explicitly. The system can be freely distributed to other users. New users are encouraged to notify the CASA coordinator so that they can be included in a user list where they will be kept up to date about the progress of the system.

Bug reports, questions and suggestions should be sent to the e-mail address casa@risc.uni-linz.ac.at.

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4 Functionality of CASA

In the system, an algebraic set—a central notion in algebraic geometry—can be represented in four different ways:

- *Implicit representation:* An algebraic set is the set of common zeros of a system of polynomial equations. To give an algebraic set in implicit form means to give finitely many polynomials.
- *Projected representation:* As a consequence of the primitive element theorem every irreducible d -dimensional algebraic set in n -dimensional space is, after a suitable linear transformation of coordinates, birationally projectable onto an irreducible d -dimensional algebraic set in $(d + 1)$ -dimensional space, which can be specified by a single polynomial in $d + 1$ variables. This can be generalized to unmixed-dimensional algebraic sets. An algebraic set in projected form is given by a polynomial and a tuple of rational functions (specifying the birational mapping).
- *Parametric representation:* Some irreducible algebraic sets can be parametrized by rational functions. An algebraic set in parametric form is given by a tuple of rational functions that parametrizes the algebraic set.
- *Representation by places:* All algebraic curves can be locally parametrized by a set of power series that are convergent around a point of the curve. An algebraic set is given by places if for each branch passing through a certain point on the algebraic set a tuple of power series that parametrizes the algebraic set around the point is specified.

The system provides a variety of operations on algebraic sets. As the efficiency of these operations is tightly bound to the way algebraic sets are represented, conversion routines have been provided to support various views on one object, to deepen the understanding of its principles, and to speed up algorithms working on algebraic sets. CASA also works with the polynomial ideals corresponding to these geometric objects.

The basic operations available in CASA include:

- ideal theoretic operations (sum, product, intersection, union)
- creating algebraic sets in different representations,
- generating curves of fixed multiplicities at given points,
- intersection, union, and difference of algebraic sets,
- computing tangent cones and tangent spaces,
- computation of the dimension of an algebraic set,
- decomposition into irreducible components,
- transformations of algebraic sets to hypersurfaces,
- computation of the singularities, genus, neighborhood graphs and adjoint curves of an algebraic curve.

Besides these basic operations, the following more advanced operations are available:

- rational parametrization of rational curves over an optimal extension field of coefficients,
- implicitization of parametrically given algebraic sets,
- Puiseux series expansions,
- multivariate resultants and Dixon's resultants
- Gröbner bases of ideals and modules,
- Gröbner walk,
- hybrid methods for finding solutions of an arbitrary system of equations,
- computation of rational points on conics,
- offset curves,
- plotting both explicitly and implicitly given curves and surfaces,
- computation of syzygy-bases.

5 Example of Application: Scientific Visualization

A powerful feature of CASA is the capability to visualize algebraic sets. For the majority of tasks, only singular or other distinguished points of algebraic sets yield some interesting information. These objects have to be treated with care, requiring special analysis. Most commonly used computer algebra systems have problems with plotting sets in implicit representation. CASA makes a thorough effort to obtain correct information about the local topology in the neighborhood of singularities and other critical points while still keeping the algorithm efficient by using hybrid symbolic-numerical methods.

The following Figures show the graphs produced by CASA for the algebraic curves defined as zeros of the following equations:

1. $\frac{93392896}{15625}x^6 + \left(\frac{94359552}{625}y^2 + \frac{(91521024}{625}y - \frac{249088}{125}x^4 + \left(\frac{1032192}{25}y^4 - 36864y^3 - \frac{7732224}{25}y^2 - 207360y + \frac{770048}{25}x^2 + (65536y^6 + 49152y^5 - 135168y^4 - 72704y^3 + 101376y^2 + 27648y - 27648\right.\right.$
2. $2x^4 - 3x^2y + y^2 - 2y^3 + y^4$
3. $y^4 - 96a^2y^2 + 100a^2x^2 - x^4$
4. $8(y^2 - x^2) - 8(y^3 + x^3) + 2y^4 + x^5$

Note that CASA also finds isolated singularities such as in curve 1.

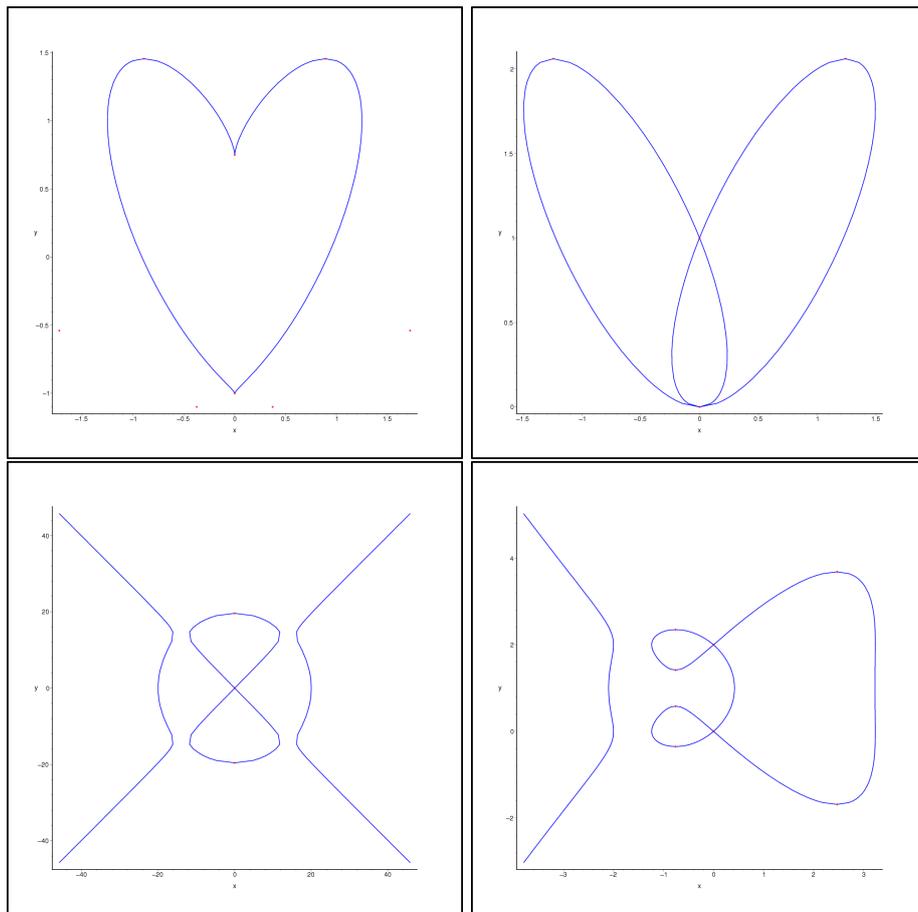


Figure 1: Plotting